

Dynamics of some higher order rational difference equations

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ABSTRACT

In this paper we discuss the solution of rational difference equation of the form $z_{n+1} = \frac{z_{n-20}}{\pm 1 \pm z_{n-6} z_{n-13} z_{n-20}}$, $n = 0, 1, \dots$ where the initial values are arbitrary real numbers. To confirm the obtained solutions we consider some numerical examples by assigning different initial values with Matlab.

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1. Introduction

In this paper we obtain solutions of rational difference Eq. 1

$$z_{n+1} = \frac{z_{n-20}}{\pm 1 \pm z_{n-6} z_{n-13} z_{n-20}}, n = 0, 1, \dots \quad (1)$$

where the initial values are arbitrary real numbers.

Difference equation is a vast field which impact almost found in every branch of pure as well as applied mathematics. In this paper we study the local stability, global attractivity of equilibrium point of Eq. 1 and boundedness of solutions of the Eq. 1). Moreover we obtain solutions of some special cases of this equation. The study and solution of nonlinear higher order difference equation is very challenging. However we have still no suitable generalized method to deal with the global behavior of rational difference equations of higher order so far. Therefore the study of rational difference equations of higher order is worth for consideration. Recently great interest is developed in studying difference equation systems. The reason is that there is need of some techniques whose can be used in investigating problems in different fields. Recently a great effort has been made in studying the qualitative analysis of rational difference equations. Difference equations are very simple in form, but it is very difficult to understand thoroughly the behaviors of their solutions (Cinar 2004a; 2004b; 2004c). Karatas et al. (2006) studied the positive solutions and attractivity of the difference equation by considering non zero

real numbers initial values $x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2} x_{n-5}}$, $n = 0, 1, \dots$ Elsayed (2008) worked on the difference equation $x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2} x_{n-5}}$, $n = 0, 1, \dots$ He found the solution of this equation and obtained graphs of numerical examples for some values of initial conditions. Elsayed (2009) investigated the difference equation by considering real numbers initial values $x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2} x_{n-5}}$, $n = 0, 1, \dots$ He checked the qualitative behavior of the difference equation. Elsayed (2010) studied the solutions of the following class of difference equation $x_{n+1} = \frac{x_{n-8}}{\pm 1 \pm x_{n-2} x_{n-5} x_{n-8}}$, $n = 0, 1, \dots$ Elsayed (2011a) investigated the rational difference equation $x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4} x_{n-9}}$, $n = 0, 1, \dots$ Elsayed (2011b) investigated the rational difference equation $x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1} x_{n-3}}$, $n = 0, 1, \dots$ In Touafek and Elsayed (2012) got the form of solutions of the rational difference systems $x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}$, $y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}$. Van Khuong and Phong (2011) investigated the difference equation $x_{n-3} = x_n(1 + x_{n-1} x_{n-2})$, $n = 0, 1, \dots$ Khaliq and Elsayed (2016) studied the solutions of some difference equations of the form $x_{n+1} = \frac{x_{n-1} x_{n-5}}{x_{n-3}(\pm 1 \pm x_{n-1} x_{n-5})}$, $n = 0, 1, \dots$

Suppose that I is some interval of real numbers and F a continuous function defined on I^{k+1} ($k+1$ copies of I), where k is some natural number. Throughout this thesis, we consider the following difference equation

$$z_{n+1} = f(z_n, z_{n-1}, \dots, z_{n-k}), n = 0, 1, \dots \quad (2)$$

for given initial values $Z_{-k}, Z_{-(k-1)}, \dots, Z_0 \in I$

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Definition 1.1 (Equilibrium Point): A point $\bar{z} \in I$ is called an equilibrium point of difference Eq. 2 if

$$\bar{z} = F(\bar{z}, \dots, \bar{z})$$

That is, $z_n = \bar{z}$ for $n \geq 0$ is a solution of difference Eq. 2.

Definition 1.2 (Periodicity): A solution $\{z_n\}_{n=-k}^{\infty}$ of Eq. 2 is called periodic with period p if there exists an integer $p \geq 1$ is a s such that $z_{n+p} = z_n$ for all $n \geq -k$. If $z_{n+p} = z_n$ holds for smallest positive integer p then solution $\{z_n\}_{n=-k}^{\infty}$ of Eq. 2 is called periodic period of prime.

Theorem 1.1: Consider the difference equation

$$z_{n+1} + a_k z_n + a_0 z_{n-k} = 0, n = 0, 1, \dots$$

where $k \in \{1, 2, \dots\}$ and a_i real numbers for all i . Then $\sum_{i=0}^k |a_i| < 1$ is a sufficient condition for the asymptotic stability of Eq. 2.

2. First equation

In this section we give a specific form of the first equation in the form

$$z_{n+1} = \frac{z_{n-20}}{\pm 1 \pm z_{n-6} z_{n-13} z_{n-20}}, n = 0, 1, \dots \tag{3}$$

with non-zero real numbers initial values.

Theorem 2.1: Let $\{z_n\}_{n=-20}^{\infty}$ be a solution of Eq. 3. Then for $n = 0, 1, \dots$

$$\begin{aligned} Z_{21n-20} &= Z_{-20} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-6} z_{-13} z_{-20}}{1+(3\alpha+1)z_{-6} z_{-13} z_{-20}} \right] = \\ w \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha g p w}{1+(3\alpha+1)g p w} \right] \\ Z_{21n-19} &= Z_{-19} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-5} z_{-12} z_{-19}}{1+(3\alpha+1)z_{-5} z_{-12} z_{-19}} \right] = \\ v \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha f \beta v}{1+(3\alpha+1)f \beta v} \right] \\ Z_{21n-18} &= Z_{-18} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-4} z_{-11} z_{-18}}{1+(3\alpha+1)z_{-4} z_{-11} z_{-18}} \right] = \\ u \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha e m u}{1+(3\alpha+1)e m u} \right] \\ Z_{21n-17} &= Z_{-17} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-3} z_{-10} z_{-17}}{1+(3\alpha+1)z_{-3} z_{-10} z_{-17}} \right] = \\ t \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha d l t}{1+(3\alpha+1)d l t} \right] \\ Z_{21n-16} &= Z_{-16} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-2} z_{-9} z_{-16}}{1+(3\alpha+1)z_{-2} z_{-9} z_{-16}} \right] = \\ s \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha c k s}{1+(3\alpha+1)c k s} \right] \\ Z_{21n-15} &= Z_{-15} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_{-1} z_{-8} z_{-15}}{1+(3\alpha+1)z_{-1} z_{-8} z_{-15}} \right] = \\ r \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha b j r}{1+(3\alpha+1)b j r} \right] \\ Z_{21n-14} &= Z_{-14} \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha z_0 z_{-7} z_{-14}}{1+(3\alpha+1)z_0 z_{-7} z_{-14}} \right] = \\ q \prod_{\alpha=0}^{n-1} \left[\frac{1+3\alpha a h q}{1+(3\alpha+1)a h q} \right] \\ Z_{21n-13} &= Z_{-13} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-20} z_{-6} z_{-13}}{1+(3\alpha+2)z_{-20} z_{-6} z_{-13}} \right] = \\ p \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)w g p}{1+(3\alpha+2)w g p} \right] \\ Z_{21n-12} &= Z_{-12} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-19} z_{-5} z_{-12}}{1+(3\alpha+2)z_{-19} z_{-5} z_{-12}} \right] = \\ \beta \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)\beta v f}{1+(3\alpha+2)\beta v f} \right] \end{aligned}$$

$$\begin{aligned} Z_{21n-11} &= Z_{-11} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-18} z_{-4} z_{-11}}{1+(3\alpha+2)z_{-18} z_{-4} z_{-11}} \right] = \\ m \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)m u e}{1+(3\alpha+2)m u e} \right] \\ Z_{21n-10} &= Z_{-10} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-17} z_{-3} z_{-10}}{1+(3\alpha+2)z_{-17} z_{-3} z_{-10}} \right] = \\ l \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)l t d}{1+(3\alpha+2)l t d} \right] \\ Z_{21n-9} &= Z_{-9} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-16} z_{-2} z_{-9}}{1+(3\alpha+2)z_{-16} z_{-2} z_{-9}} \right] = \\ k \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)k s c}{1+(3\alpha+2)k s c} \right] \\ Z_{21n-8} &= Z_{-8} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-15} z_{-1} z_{-8}}{1+(3\alpha+2)z_{-15} z_{-1} z_{-8}} \right] = \\ j \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)b j r}{1+(3\alpha+2)b j r} \right] \\ Z_{21n-7} &= Z_{-7} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)z_{-14} z_0 z_{-7}}{1+(3\alpha+2)z_{-14} z_0 z_{-7}} \right] = \\ h \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)a h q}{1+(3\alpha+2)a h q} \right] \\ Z_{21n-6} &= Z_{-6} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-13} z_{-20} z_{-6}}{1+(3\alpha+3)z_{-13} z_{-20} z_{-6}} \right] = \\ g \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)g p w}{1+(3\alpha+3)g p w} \right] \\ Z_{21n-5} &= Z_{-5} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-12} z_{-19} z_{-5}}{1+(3\alpha+3)z_{-12} z_{-19} z_{-5}} \right] = \\ f \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)f \beta v}{1+(3\alpha+3)f \beta v} \right] \\ Z_{21n-4} &= Z_{-4} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-11} z_{-18} z_{-4}}{1+(3\alpha+3)z_{-11} z_{-18} z_{-4}} \right] = \\ e \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)e m u}{1+(3\alpha+3)e m u} \right] \\ Z_{21n-3} &= Z_{-3} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-10} z_{-17} z_{-3}}{1+(3\alpha+3)z_{-10} z_{-17} z_{-3}} \right] = \\ d \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)d l t}{1+(3\alpha+3)d l t} \right] \\ Z_{21n-2} &= Z_{-2} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-9} z_{-16} z_{-2}}{1+(3\alpha+3)z_{-9} z_{-16} z_{-2}} \right] = \\ c \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)c k s}{1+(3\alpha+3)c k s} \right] \\ Z_{21n-1} &= Z_{-1} \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-8} z_{-15} z_{-1}}{1+(3\alpha+3)z_{-8} z_{-15} z_{-1}} \right] = \\ b \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)b j r}{1+(3\alpha+3)b j r} \right] \\ Z_{21n} &= Z_0 \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)z_{-7} z_{-14} z_0}{1+(3\alpha+3)z_{-7} z_{-14} z_0} \right] = \\ a \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+2)a h q}{1+(3\alpha+3)a h q} \right] \end{aligned}$$

where

$$\begin{aligned} z_{-20} &= w, z_{-19} = v, z_{-18} = u, z_{-17} = t, z_{-16} = s, \\ z_{-15} &= r, z_{-14} = q, z_{-13} = p, z_{-12} = \beta, z_{-11} = m, z_{-10} = l, \\ z_{-9} &= k, z_{-8} = j, z_{-7} = h, z_{-6} = g, z_{-5} = f, z_{-4} = e, \\ z_{-3} &= d, z_{-2} = c, z_{-1} = b, z_0 = a, \end{aligned}$$

Proof: For $n = 0$ the result follow. Suppose that $n > 0$ and that our assumption is true for $n - 1$. That is

$$\begin{aligned} Z_{21n-41} &= w \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha g p w}{1+(3\alpha+1)g p w} \right], Z_{21n-40} = \\ v \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha f \beta v}{1+(3\alpha+1)f \beta v} \right], \\ Z_{21n-39} &= u \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha e m u}{1+(3\alpha+1)e m u} \right], Z_{21n-38} = \\ t \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha d l t}{1+(3\alpha+1)d l t} \right], \\ Z_{21n-37} &= s \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha c k s}{1+(3\alpha+1)c k s} \right], Z_{21n-36} = \\ r \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha b j r}{1+(3\alpha+1)b j r} \right] \\ Z_{21n-35} &= q \prod_{\alpha=0}^{n-2} \left[\frac{1+3\alpha a h q}{1+(3\alpha+1)a h q} \right], Z_{21n-34} = \\ p \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)w g p}{1+(3\alpha+2)w g p} \right], \\ Z_{21n-33} &= \beta \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)\beta v f}{1+(3\alpha+2)\beta v f} \right], Z_{21n-32} = \\ m \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)m u e}{1+(3\alpha+2)m u e} \right], \\ Z_{21n-31} &= l \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)l t d}{1+(3\alpha+2)l t d} \right], Z_{21n-30} = \\ k \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)k s c}{1+(3\alpha+2)k s c} \right] \end{aligned}$$

$$\begin{aligned}
 Z_{21n-29} &= j \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)bjr}{1+(3\alpha+2)bjr} \right], Z_{21n-28} = \\
 &h \prod_{\alpha=0}^{n-1} \left[\frac{1+(3\alpha+1)ahq}{1+(3\alpha+2)ahq} \right], \\
 Z_{21n-27} &= g \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)gpw}{1+(3\alpha+3)gpw} \right], Z_{21n-26} = \\
 &f \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)f\beta v}{1+(3\alpha+3)f\beta v} \right], \\
 Z_{21n-25} &= e \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)emu}{1+(3\alpha+3)emu} \right], Z_{21n-24} = \\
 &d \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)dlt}{1+(3\alpha+3)dlt} \right] \\
 Z_{21n-23} &= c \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)cks}{1+(3\alpha+3)cks} \right], Z_{21n-22} = \\
 &b \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)bjr}{1+(3\alpha+3)bjr} \right], \\
 Z_{21n-21} &= a \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)ahq}{1+(3\alpha+3)ahq} \right]
 \end{aligned}$$

now it follows from Eq. 3

$$\begin{aligned}
 Z_{21n-20} &= \frac{Z_{21n-41}}{1+Z_{21n-27}Z_{21n-34}Z_{21n-41}} \\
 &= \frac{w \prod_{\alpha=0}^{n-2} \left[\frac{1+3agpw}{1+(3\alpha+1)gpw} \right]}{1+gpw \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+2)gpw}{1+(3\alpha+3)gpw} \right] \prod_{\alpha=0}^{n-2} \left[\frac{1+(3\alpha+1)gpw}{1+(3\alpha+2)gpw} \right] w \prod_{\alpha=0}^{n-2} \left[\frac{1+3agpw}{1+(3\alpha+1)gpw} \right]} \\
 &= \frac{w \prod_{\alpha=0}^{n-2} \left[\frac{1+3agpw}{1+(3\alpha+1)gpw} \right]}{1+gpw \prod_{\alpha=0}^{n-2} \left[\frac{1+3agpw}{1+(3\alpha+3)gpw} \right]} \\
 &= \frac{w \prod_{\alpha=0}^{n-2} \left[\frac{1+3agpw}{1+(3\alpha+1)gpw} \right]}{\frac{1+(3\alpha-2)gpw}{1+(3\alpha-3)gpw}} \\
 Z_{21n-20} &= w \prod_{\alpha=0}^{n-1} \left[\frac{1+3agpw}{1+(3\alpha+1)gpw} \right]
 \end{aligned}$$

Similarly, other relations can be proved in same manner.

Theorem 2.2: Eq. 3 has zero as equilibrium point.

Proof: To find the equilibrium points of Eq. 3

$$\begin{aligned}
 \bar{z} &= \frac{\bar{z}}{1+\bar{z}^3} \\
 \bar{z}^4 &= 0
 \end{aligned}$$

thus $\bar{z} = 0$ is the equilibrium point of Eq. 3.

Theorem 2.3: For Eq. 3, every positive solution is bounded.

Proof: Let $\{z_n\}_{n=-20}^{\infty}$ be a solution of Eq. 3. Then from Eq. 3

$$z_{n+1} = \frac{z_{n-20}}{1+z_{n-6}z_{n-13}z_{n-20}} \leq z_{n-20}$$

then $z_{n+1} \leq z_{n-20}$ for all $n \geq 0$. Then the sequence $\{z_n\}_{n=-20}^{\infty}$ is decreasing and are bounded from above by

$$M = \max \{ Z_{-20}, Z_{-19}, Z_{-18}, Z_{-17}, Z_{-16}, Z_{-15}, Z_{-14}, Z_{-13}, Z_{-12}, Z_{-11}, Z_{-10}, \{ Z_{-9}, Z_{-8}, Z_{-7}, Z_{-6}, Z_{-5}, Z_{-4}, Z_{-3}, Z_{-2}, Z_{-1}, Z_{-0}, \}$$

2.1. Numerical examples

For confirming the results, suppose some numerical examples which show different types of solutions of Eq. 3.

Example 2.1: Assume that (Fig. 1)

$$\begin{aligned}
 z_{-20} &= 15, z_{-19} = -2.5, z_{-18} = -3, z_{-17} = 2, z_{-16} = \\
 &-6, z_{-15} = 3, z_{-14} = -7, \\
 z_{-13} &= 8, z_{-12} = 0, z_{-11} = -10, z_{-10} = 9, z_{-9} = \\
 &-3.5, z_{-8} = 9, z_{-7} = 15, \\
 z_{-6} &= 5.5, z_{-5} = 2, z_{-4} = 0, z_{-3} = 13, z_{-2} = 11, z_{-1} = \\
 &8.5, z_0 = 1
 \end{aligned}$$

Example 2.2: Assume that (Fig. 2)

$$\begin{aligned}
 z_{-20} &= 10, z_{-19} = 3, z_{-18} = 10, z_{-17} = -5, z_{-16} = \\
 &-6, z_{-15} = 9, z_{-14} = -7.5, \\
 z_{-13} &= 10, z_{-12} = 0, z_{-11} = -15, z_{-10} = 9, z_{-9} = \\
 &-3.5, z_{-8} = 3.5, z_{-7} = 15, z_{-6} = 10.5, \\
 z_{-5} &= 7, z_{-4} = 0, z_{-3} = 18, z_{-2} = 11, z_{-1} = 9.5, z_0 = 6
 \end{aligned}$$

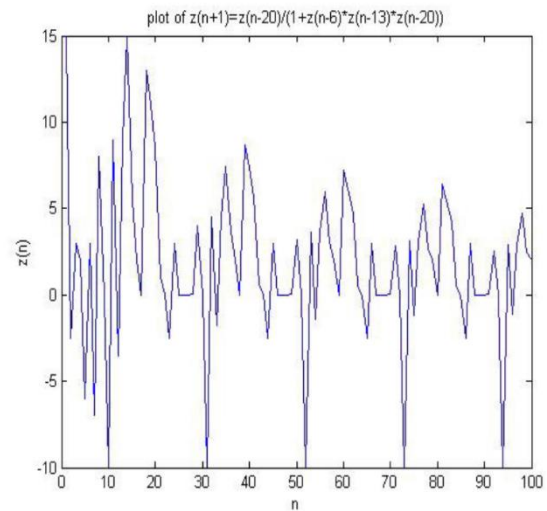


Fig. 1: $z(n+1) = z(n-20)/(-1+z(n-6)z(n-13)z(n-20))$

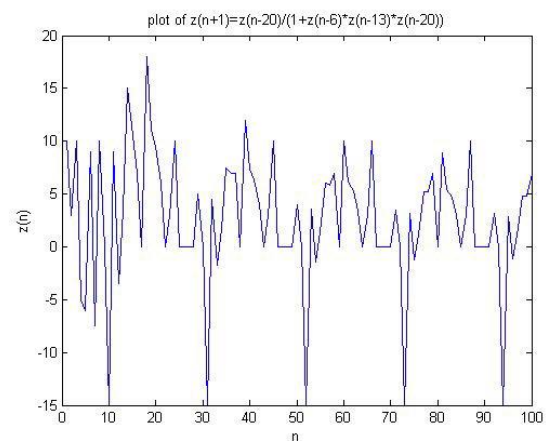


Fig. 2: $z(n+1) = z(n-20)/(-1+z(n-6)z(n-13)z(n-20))$

3. Second equation

In this portion we give a specific form of the equation in the form

$$z_{n+1} = \frac{z_{n-20}}{-1+z_{n-6}z_{n-13}z_{n-20}}, n \in N, n = 0, 1, \dots, \quad (4)$$

with non-zero real numbers initial values and

$$z_{\alpha-6}z_{\alpha-13}z_{\alpha-20} \neq 1$$

for

$$(\alpha = 0,1,2,3,4,5,6)$$

Theorem 3.1: Every solution $\{z_n\}_{n=-20}^{\infty}$ of Eq. 4 is periodic with period 42 and is of the form

$$\left\{ \begin{array}{l} w, v, u, t, s, r, q, p, \beta, m, l, k, j, h, g, f, e, d, c, b, a, \frac{w}{-1+gpw}, \\ \frac{v}{-1+f\beta v}, \frac{u}{-1+emu}, \frac{t}{-1+dlt}, \frac{s}{-1+cks}, \frac{r}{-1+bjr}, \frac{q}{-1+ahq} \\ p(-1+wg p), \beta(-1+vf\beta), m(-1+emu), l(-1+dlt), \\ k(-1+cks), j(-1+bjr), h(-1+ahq), \frac{g}{-1+gpw}, \frac{f}{-1+vf\beta} \\ \frac{e}{-1+emu}, \frac{d}{1+dlt}, \frac{c}{-1+cks}, \frac{b}{-1+bjr}, \frac{a}{-1+ahq}, w, v, u \end{array} \right\}$$

Proof: From Eq. 4 it is clear that

$$\begin{aligned} z_1 &= \frac{w}{-1+gpw}, z_2 = \frac{v}{-1+f\beta v}, z_3 = \frac{u}{-1+emu}, z_4 = \frac{t}{-1+dlt} \\ z_5 &= \frac{s}{-1+cks}, z_6 = \frac{r}{-1+bjr}, z_7 = \frac{q}{-1+ahq}, z_8 = p(-1+wg p), \\ z_9 &= \beta(-1+vf\beta), z_{10} = m(-1+emu), z_{11} = \\ & l(-1+dlt), z_{12} = k(-1+cks), \\ z_{13} &= j(-1+bjr), z_{14} = h(-1+ahq), z_{15} = \frac{g}{-1+gpw}, z_{16} = \\ & \frac{f}{-1+vf\beta}, \\ z_{17} &= \frac{e}{-1+emu}, z_{18} = \frac{d}{-1+dlt}, z_{19} = \frac{c}{-1+cks}, z_{20} = \\ & \frac{b}{-1+bjr}, z_{21} = \frac{a}{-1+ahq}. \end{aligned}$$

Theorem 3.2: Eq. 4 has two equilibrium points 0 and $\sqrt[3]{2}$ which are not locally asymptotically stable.

Proof: To find equilibrium point from Eq. 4

$$\begin{aligned} \bar{z} &= \frac{\bar{z}}{-1+\bar{z}^3} \\ \bar{z} &= 0, \bar{z} = \sqrt[3]{2} \end{aligned}$$

$$\left\{ \begin{array}{l} w = \frac{w}{-1+gpw}, v = \frac{v}{-1+f\beta v}, u = \frac{u}{-1+emu}, t = \frac{t}{-1+dlt}, \\ s = \frac{s}{-1+cks}, r = \frac{r}{-1+bjr}, q = \frac{q}{-1+ahq}, p = p(-1+wg p), \\ \beta = \beta(-1+vf\beta), m = m(-1+emu), l = l(-1+dlt), k = k(-1+cks) \\ j = j(-1+bjr), h = h(-1+ahq), g = \frac{g}{-1+gpw}, f = \frac{f}{-1+vf\beta} \\ e = \frac{e}{-1+emu}, d = \frac{d}{1+dlt}, c = \frac{c}{-1+cks}, b = \frac{b}{-1+bjr}, a = \frac{a}{-1+ahq}. \end{array} \right\}$$

assume that $ahq = bj r = cks = dlt = emu = vf\beta = gpw = 2$, then we see from conditions of above theorem

$$\begin{aligned} z_{42n-20} &= w, z_{42n-19} = v, z_{42n-18} = u, z_{42n-17} = \\ & t, z_{42n-16} = s, z_{42n-15} = r, \\ z_{42n-14} &= q, z_{42n-13} = p, z_{42n-12} = \beta, z_{42n-11} = \\ & m, z_{42n-10} = l, z_{42n-9} = k, \\ z_{42n-8} &= j, z_{42n-7} = h, z_{42n-6} = g, z_{42n-5} = f, z_{42n-4} = \\ & e, z_{42n-3} = d, \\ z_{42n-2} &= c, z_{42n-1} = b, z_{42n} = a. \end{aligned}$$

Thus solution is of period 21.

3.1. Numerical examples

For confirming the results, suppose some numerical examples which shows different types of solutions of Eq. 4.

Let $f: (0, \infty)^3 \rightarrow (0, \infty)$ be a function defined by

$$\begin{aligned} f(w, x, y) &= \frac{y}{-1+wxy} \\ f_w(w, x, y) &= \frac{-xy^2}{(-1+wxy)^2} \\ f_x(w, x, y) &= \frac{-wy^2}{(-1+wxy)^2} \\ f_y(w, x, y) &= \frac{-1}{(-1+wxy)^2} \end{aligned}$$

At $\bar{z} = 0$

$$f_w(\bar{z}, \bar{z}, \bar{z}) = 0, f_x(\bar{z}, \bar{z}, \bar{z}) = 0, f_y(\bar{z}, \bar{z}, \bar{z}) = -1.$$

So linearized equation of Eq. 4 about $\bar{z} = 0$ is $y_{n+1} + y_{n-2} = 0$
at $\bar{z} = \sqrt[3]{2}$

$$f_w(\bar{z}, \bar{z}, \bar{z}) = -2, f_x(\bar{z}, \bar{z}, \bar{z}) = -2, f_y(\bar{z}, \bar{z}, \bar{z}) = -1.$$

So linearized equation of Eq. 4 about $\bar{z} = \sqrt[3]{2}$ is

$$y_{n+1} + 2y_n + y_{n-2} = 0.$$

By the generalization of theorem [A] we find that $\bar{z} = 0, \bar{z} = \sqrt[3]{2}$ are unstable.

Theorem 3.3: Eq. 4 has a periodic solution of period 21 if $ahq = bj r = cks = dlt = emu = vf\beta = gpw = 2$, and then takes the form

$$\{w, v, u, t, s, r, q, p, \beta, m, l, k, j, h, g, f, e, d, c, b, a, w, v, u, \dots\}$$

Proof: First assume that there is a prime period 21 solution from conditions of above theorem

Example 3.1: Assume that (Fig. 3)

$$\begin{aligned} z_{-10} &= 10, z_{-19} = -18.5, z_{-18} = -5, z_{-17} = 10, z_{-16} = \\ & 13.5, z_{-15} = -8.5, z_{-14} = 1.5, \\ z_{-13} &= 1.2, z_{-12} = -9, z_{-11} = 15, z_{-10} = -7, z_{-9} = \\ & 6, z_{-8} = 17, z_{-7} = 19.5, z_{-6} = 2, \\ z_{-5} &= 9.5, z_{-4} = 16.5, z_{-3} = 11.5, z_{-2} = 14, z_{-1} = \\ & 10.5, z_0 = -5.5 \end{aligned}$$

Example 3.2: Assume that (Fig. 4)

$$\begin{aligned} z_{-20} &= 15, z_{-19} = -2.5, z_{-18} = -3, z_{-17} = 2, z_{-16} = \\ & -6, z_{-15} = 3, z_{-14} = -7, \\ z_{-13} &= 8, z_{-12} = 0, z_{-11} = -10, z_{-10} = 9, z_{-9} = \\ & -3.5, z_{-8} = 9, z_{-7} = 15, z_{-6} = 5.5, \\ z_{-5} &= 2, z_{-4} = 0, z_{-3} = 13, z_{-2} = 114, z_{-1} = 8.5, z_0 = 1 \end{aligned}$$

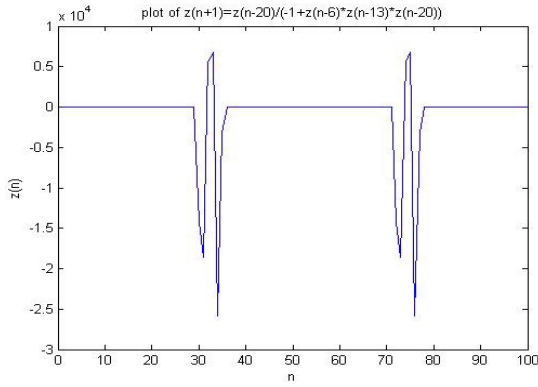


Fig. 3: $z(n+1) = z(n-20)/(-1+z(n-6)z(n-13)z(n-20))$

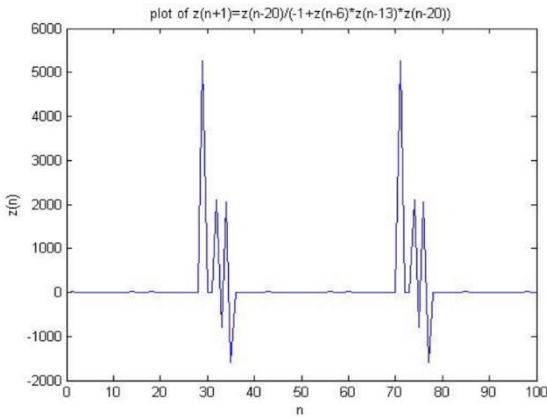


Fig. 4: $z(n+1) = z(n-20)/(-1+z(n-6)z(n-13)z(n-20))$

4. Third equation

In this part we give a specific form of equation in the form

$$z_{n+1} = \frac{z_{n-20}}{1-z_{n-6}z_{n-13}z_{n-20}}, n = 0,1, \dots \tag{5}$$

with non-zero real numbers initial values and

$$z_{\alpha-6}z_{\alpha-13}z_{\alpha-20} \neq 1$$

for

$$(\alpha = 0,1,2,3,4,5,6)$$

Theorem 4.1: Let $\{z_n\}_{n=-20}^{\infty}$ be a solution of Eq. 5. Then for $n = 0,1, \dots$

$$\begin{aligned} Z_{21n-20} &= Z_{-20} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-6} z_{-13} z_{-20}}{1-(3\alpha+1)z_{-6}z_{-13}z_{-20}} \right] = \\ &w \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha gpw}{1-(3\alpha+1)gpw} \right] \\ Z_{21n-19} &= Z_{-19} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-5} z_{-12} z_{-19}}{1-(3\alpha+1)z_{-5}z_{-12}z_{-19}} \right] = \\ &v \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha f \beta v}{1-(3\alpha+1)f \beta v} \right] \\ Z_{21n-18} &= Z_{-18} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-4} z_{-11} z_{-18}}{1-(3\alpha+1)z_{-4}z_{-11}z_{-18}} \right] = \\ &u \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha emu}{1-(3\alpha+1)emu} \right] \\ Z_{21n-17} &= Z_{-17} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-3} z_{-10} z_{-17}}{1-(3\alpha+1)z_{-3}z_{-10}z_{-17}} \right] = \\ &t \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha dlt}{1-(3\alpha+1)dlt} \right] \end{aligned}$$

$$\begin{aligned} Z_{21n-16} &= Z_{-16} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-2} z_{-9} z_{-16}}{1-(3\alpha+1)z_{-2}z_{-9}z_{-16}} \right] = \\ &s \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha cks}{1-(3\alpha+1)cks} \right] \\ Z_{21n-15} &= Z_{-15} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_{-1} z_{-8} z_{-15}}{1-(3\alpha+1)z_{-1}z_{-8}z_{-15}} \right] = \\ &r \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha bjr}{1-(3\alpha+1)bjr} \right] \\ Z_{21n-14} &= Z_{-14} \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha z_0 z_{-7} z_{-14}}{1-(3\alpha+1)z_0z_{-7}z_{-14}} \right] = \\ &q \prod_{\alpha=0}^{n-1} \left[\frac{1-3\alpha ahq}{1-(3\alpha+1)ahq} \right] \\ Z_{21n-13} &= Z_{-13} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-20} z_{-6} z_{-13}}{1-(3\alpha+2)z_{-20}z_{-6}z_{-13}} \right] = \\ &p \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)wgp}{1-(3\alpha+2)wgp} \right] \\ Z_{21n-12} &= Z_{-12} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-19} z_{-5} z_{-12}}{1-(3\alpha+2)z_{-19}z_{-5}z_{-12}} \right] = \\ &\beta \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)\beta v f}{1-(3\alpha+2)\beta v f} \right] \\ Z_{21n-11} &= Z_{-11} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-18} z_{-4} z_{-11}}{1-(3\alpha+2)z_{-18}z_{-4}z_{-11}} \right] = \\ &m \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)mue}{1-(3\alpha+2)mue} \right] \\ Z_{21n-10} &= Z_{-10} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-17} z_{-3} z_{-10}}{1-(3\alpha+2)z_{-17}z_{-3}z_{-10}} \right] = \\ &l \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)ltd}{1-(3\alpha+2)ltd} \right] \\ Z_{21n-9} &= Z_{-9} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-16} z_{-2} z_{-9}}{1-(3\alpha+2)z_{-16}z_{-2}z_{-9}} \right] = \\ &k \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)ksc}{1-(3\alpha+2)ksc} \right] \\ Z_{21n-8} &= Z_{-8} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-15} z_{-1} z_{-8}}{1-(3\alpha+2)z_{-15}z_{-1}z_{-8}} \right] = \\ &j \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)bjr}{1-(3\alpha+2)bjr} \right] \\ Z_{21n-7} &= Z_{-7} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)z_{-14} z_0 z_{-7}}{1-(3\alpha+2)z_{-14}z_0z_{-7}} \right] = \\ &h \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+1)ahq}{1-(3\alpha+2)ahq} \right] \\ Z_{21n-6} &= Z_{-6} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-13} z_{-20} z_{-6}}{1-(3\alpha+3)z_{-13}z_{-20}z_{-6}} \right] = \\ &g \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)gpw}{1-(3\alpha+3)gpw} \right] \\ Z_{21n-5} &= Z_{-5} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-12} z_{-19} z_{-5}}{1-(3\alpha+3)z_{-12}z_{-19}z_{-5}} \right] = \\ &f \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)f \beta v}{1-(3\alpha+3)f \beta v} \right] \\ Z_{21n-4} &= Z_{-4} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-11} z_{-18} z_{-4}}{1-(3\alpha+3)z_{-11}z_{-18}z_{-4}} \right] = \\ &e \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)emu}{1-(3\alpha+3)emu} \right] \\ Z_{21n-3} &= Z_{-3} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-10} z_{-17} z_{-3}}{1-(3\alpha+3)z_{-10}z_{-17}z_{-3}} \right] = \\ &d \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)dlt}{1-(3\alpha+3)dlt} \right] \\ Z_{21n-2} &= Z_{-2} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-9} z_{-16} z_{-2}}{1-(3\alpha+3)z_{-9}z_{-16}z_{-2}} \right] = \\ &c \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)cks}{1-(3\alpha+3)cks} \right] \\ Z_{21n-1} &= Z_{-1} \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-8} z_{-15} z_{-1}}{1-(3\alpha+3)z_{-8}z_{-15}z_{-1}} \right] = \\ &b \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)bjr}{1-(3\alpha+3)bjr} \right] \\ Z_{21n} &= Z_0 \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)z_{-7} z_{-14} z_0}{1-(3\alpha+3)z_{-7}z_{-14}z_0} \right] = a \prod_{\alpha=0}^{n-1} \left[\frac{1-(3\alpha+2)ahq}{1-(3\alpha+3)ahq} \right] \end{aligned}$$

where

$$\begin{aligned} z_{-20} &= w, z_{-19} = v, z_{-18} = u, z_{-17} = t, z_{-16} = s, z_{-15} = \\ &r, z_{-14} = q, z_{-13} = p, z_{-12} = \beta, \\ &z_{-11} = m, z_{-10} = l, z_{-9} = k, z_{-8} = j, z_{-7} = h, z_{-6} = g, z_{-5} = \\ &f, z_{-4} = e, z_{-3} = d, z_{-2} = c, \\ &z_{-1} = b, z_0 = a \end{aligned}$$

and

$$z_{\alpha-6}z_{\alpha-13}z_{\alpha-20} \neq 1$$

for

$(\alpha = 0,1,2,3,4,5,6)$

Proof: The proof is similar to the proof of theorem 2.1.

Remark 4.1: The equilibrium point of Eq. 5 is zero which is not asymptotically stable.

4.1. Numerical examples

For confirming the results, take numerical examples which show different types of solutions of Eq. 5.

Example 4.1: Assume that (Fig 5)

$z_{-20} = 18, z_{-19} = 10, z_{-18} = -5, z_{-17} = 0, z_{-16} = 11, z_{-15} = 2.5,$
 $z_{-14} = -1.5, z_{-13} = 7.7, z_{-12} = 9, z_{-11} = -15, z_{-10} = -3, z_{-9} = 7,$
 $z_{-8} = 10, z_{-7} = 17.5, z_{-6} = 2, z_{-5} = 15.5, z_{-4} = 16.5, z_{-3} = 3.5,$
 $z_{-2} = 17, z_{-1} = 11.5, z_0 = -10$

Example 4.2: Assume that (Fig 6)

$z_{-20} = 10, z_{-19} = -18.5, z_{-18} = -5, z_{-17} = 10, z_{-16} = 13.5, z_{-15} = 8.5,$
 $z_{-14} = 1.5, z_{-13} = 1.2, z_{-12} = -9, z_{-11} = 15, z_{-10} = -7, z_{-9} = 6,$
 $z_{-8} = 17, z_{-7} = 19.5, z_{-6} = 2, z_{-5} = 9.5, z_{-4} = 16.5, z_{-3} = 11.5,$
 $z_{-2} = 14, z_{-1} = 10.5, z_0 = -5.5$

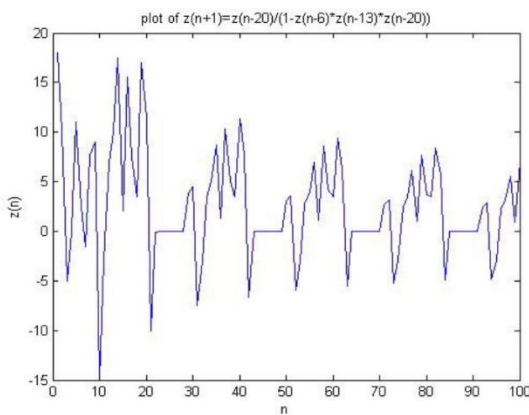


Fig. 5: $z(n + 1) = z(n - 20) / (1 - z(n - 6)z(n - 13)z(n - 20))$

5. Fourth equation

In this part we give solution of the equation in the form

$$z_{n+1} = \frac{z_{n-20}}{-1-z_{n-6}z_{n-13}z_{n-20}}, n = 0,1, \dots, \tag{6}$$

with non-zero real numbers initial values and

$$z_{\alpha-6}z_{\alpha-13}z_{\alpha-20} \neq 1$$

for

$(\alpha = 0,1,2,3,4,5,6)$

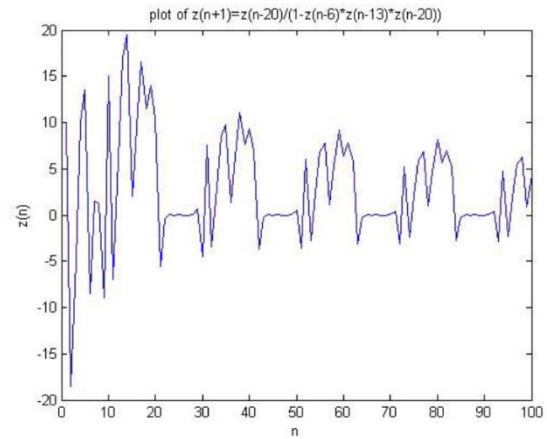


Fig. 6: $z(n + 1) = z(n - 20) / (1 - z(n - 6)z(n - 13)z(n - 20))$

Theorem 5.1: Every solution $\{z_n\}_{n=-20}^{\infty}$ of Eq. 6 is periodic with period 42 and is of the form

$$\left\{ \begin{array}{l} w, v, u, t, s, r, q, p, \beta, m, l, k, j, h, g, f, e, d, c, b, a, \frac{-w}{1+gpw}, \\ \frac{-v}{1+f\beta v}, \frac{-u}{1+emu}, \frac{-t}{1+dlt}, \frac{-s}{1+cks}, \frac{-r}{1+bjr}, \frac{-q}{1+ahq} \\ -p(1+wgp), -\beta(1+vf\beta), -(1+emu), -l(1+dlt) \\ -k(1+cks), -j(1+bjr), -h(1+ahq), \frac{-g}{1+gpw}, \frac{-f}{1+vf\beta} \\ \frac{-e}{1+emu}, \frac{-d}{1+dlt}, \frac{-c}{1+cks}, \frac{-b}{1+bjr}, \frac{-a}{1+ahq}, w, v, u \end{array} \right\}$$

Proof: The proof is similar to the proof of the theorem 3.1

Remark 5.1: Eq. 6 has two equilibrium points 0 which is not locally asymptotically stable.

Remark 5.2: Eq. 6 has a periodic solution of period 21 if

$$ahq = bj r = cks = dlt = emu = vf \beta = gpw = -2$$

and then takes the form

$$w, v, u, t, s, r, q, p, \beta, m, l, k, j, h, g, f, e, d, c, b, a, w, v, u, \dots$$

5.1. Numerical examples

For confirming the results, consider numerical examples which represent different kinds of solutions of Eq. 6.

Example 5.1: Assume that (Fig 7)

$z_{-20} = 10, z_{-19} = -8.5, z_{-18} = -6, z_{-17} = 9, z_{-16} = -12, z_{-15} = 0, z_{-14} = -9.5,$
 $z_{-13} = 9.5, z_{-12} = 3.5, z_{-11} = -10, z_{-10} = 16.5, z_{-9} = 3.5, z_{-8} = -5, z_{-7} = 18, z_{-6} = -5.5,$
 $z_{-5} = 2, z_{-4} = 0.5, z_{-3} = -15, z_{-2} = 12.5, z_{-1} = 8.5, z_0 = -1$

Example 5.2: Assume that (Fig 8)

$z_{-20} = 15, z_{-19} = -2.5, z_{-18} = -3, z_{-17} = 2, z_{-16} = -9, z_{-15} = 3, z_{-14} = -7,$
 $z_{-13} = 0.8, z_{-12} = 0, z_{-11} = -18, z_{-10} = 9, z_{-9} = -3.5, z_{-8} = 9, z_{-7} = 15,$
 $z_{-6} = 5.5, z_{-5} = 2, z_{-4} = 0, z_{-3} = 13, z_{-2} = 1.1, z_{-1} = 8.5, z_0 = 11$

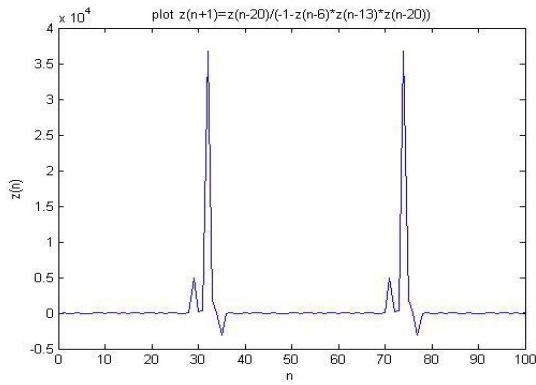


Fig. 7: $z(n + 1) = z(n - 20) / (1 - z(n - 6)z(n - 13)z(n - 20))$

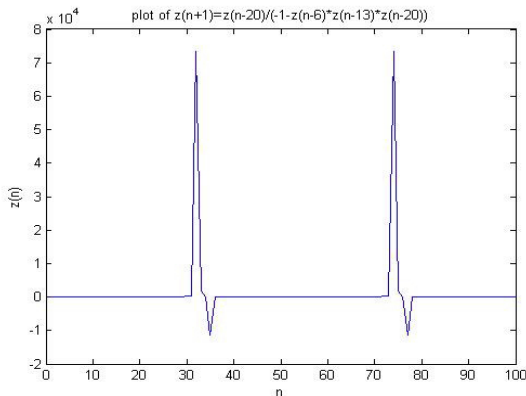


Fig. 8: $z(n + 1) = z(n - 20) / (-1 - z(n - 6)z(n - 13)z(n - 20))$

6. Conclusion

In this paper we studied solutions, equilibrium points and periodicity of four types of difference equations of Eq. 1. Eq. 3 has zero as equilibrium point and every positive solution is bounded. Eq. 4 has two equilibrium points 0 and $\sqrt[3]{2}$ which are not locally asymptotically stable and has a periodic solution of period 42. The equilibrium point of Eq. 5 is zero which is not asymptotically stable. Every solution of Eq. 6 is periodic with period 21 and has two equilibrium points 0 which is not locally

asymptotically stable. To confirm the obtained result, we gave numerical examples of each case by using Matlab.

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